

Equipartition interpretation of the Koide relation: representation-theoretic formulation, covariance test, and neutrino mass predictions

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The charged-lepton masses satisfy the Koide relation $Q \equiv (\sum m)/(\sum \sqrt{m})^2 = 2/3$ to a relative accuracy of 3×10^{-6} , without an accepted explanation for four decades. We show that in the variables $\chi_i = \sqrt{m_i}$ the relation is exactly *equipartition of the norm between the singlet and doublet irreducible representations of the family group S_3* , and prove that the unique S_3 -invariant potential of degree ≤ 4 enforcing this as a minimum at every scale is $V_{\text{eq}} = (n_1 - n_2)^2$. The ratio Q is a renormalization-group quasi-invariant, exact in the on-shell scheme with a closed-form $O(\alpha)$ drift in $\overline{\text{MS}}$. Interpreting interactions as covariances between modes predicts that deviations from $2/3$ grow with the coupling: across fermion triplets $|Q - \frac{2}{3}|$ rises monotonically over five orders of magnitude, with signs reproduced by a level-repulsion formula and fitted correlations matching $C_F \alpha_s / \pi$ in the perturbative regime. Extended to neutrinos through the hyperoctahedral group B_3 , the equipartition cone intersects exactly one sign orbit - a single reflection on the lightest mode - yielding, for normal ordering, $\Sigma m_\nu = 59.16 \pm 0.24$ meV, $m_\beta = 8.85$ meV and $m_{\beta\beta} = 1.3 - 4.0$ meV: a sharply falsifiable set of predictions for JUNO, cosmology and $0\nu\beta\beta$ searches. The local statistical significance of the relation is $\sim 4.3\sigma$, robust under look-elsewhere accounting, and independently supported by the trials-free 1981 out-of-sample prediction of the τ mass.

I. INTRODUCTION

The charged-lepton masses satisfy the empirical relation of Koide [1, 2],

$$Q \equiv \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3}, \quad (1)$$

which with current data holds to a relative accuracy of 3.3×10^{-6} : $Q = 0.6666645 \pm 0.0000051$, the uncertainty being dominated entirely by the τ mass [3]. Remarkably, the relation was proposed when the measured τ mass, $m_\tau = 1784.2 \pm 3.2$ MeV, placed it more than 2σ away from $2/3$; Eq. (1) effectively predicted $m_\tau = 1776.97$ MeV, where subsequent measurements landed [3, 4].

Despite this precision and its prospective success, Eq. (1) has resisted explanation. Standard objections are threefold: the relation holds for pole masses and drifts for running masses; naive extensions to quark triplets fail by large amounts [5]; and, as a single number, it may be a numerological coincidence. In this paper we address all three within a single framework and extract new, falsifiable predictions from it.

Our starting point is a change of variables: we treat $\chi_i \equiv \sqrt{m_i}$ as the fundamental quantities, with the mass playing the role of an additive, variance-like norm $m_i = \chi_i^2$. The contributions of this work are then: (i) a representation-theoretic reformulation - Eq. (1) is equipartition of $|\vec{\chi}|^2$ between the irreducible representations of the family permutation group S_3 - together

with a uniqueness theorem for the invariant potential enforcing it (Sec. II); (ii) a renormalization-group analysis showing that Q is a quasi-invariant, exact in the on-shell scheme, with the $\overline{\text{MS}}$ drift computed in closed form (Sec. III); (iii) a quantitative *covariance test*: if interactions are covariances between modes, deviations from $2/3$ must grow with the coupling - verified across fermion triplets over five orders of magnitude, including signs (Sec. IV); (iv) an extension to neutrinos via the hyperoctahedral group B_3 , which uniquely selects the sign orbit of Brannen's construction and yields a complete, testable set of neutrino observables (Sec. V); (v) a statistical significance analysis with explicit look-elsewhere accounting and a trials-free out-of-sample test (Sec. VI); and (vi) robustness checks and future-sensitivity estimates (Sec. VII). We conclude in Sec. VIII.

Throughout, charged-lepton and top masses are pole masses and quark masses are $\overline{\text{MS}}$ values at their conventional scales [3]; the role of this choice is itself part of the analysis (Secs. III and VIII).

II. EQUIPARTITION FORMULATION

A. Irreducible decomposition and the 45° cone

Let $\chi_i \equiv \sqrt{m_i}$. Under the permutation group S_3 the vector $\vec{\chi} \in \mathbb{R}^3$ decomposes as $\mathbf{1} \oplus \mathbf{2}$, with singlet norm $n_1 = (\sum_i \chi_i)^2/3$ and doublet norm $n_2 = \sum_i \chi_i^2 - n_1$. Then

$$n_1 = n_2 \iff Q = \frac{2}{3}, \quad (2)$$

i.e. the Koide relation is *equipartition of the norm between the irreducible representations*, equivalently a 45°

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angle between $\vec{\chi}$ and the democratic axis (first noted geometrically in [6]). The measured angle is $\theta = 44.99991^\circ \pm 0.00022^\circ$. Reference points: $Q = \frac{1}{3}$ (degenerate, pure singlet), $Q \rightarrow 1$ (maximal hierarchy, *per-dimension* equipartition $\frac{1}{3} : \frac{2}{3}$); nature sits at *per-representation* equipartition $\frac{1}{2} : \frac{1}{2}$.

B. Uniqueness of the enforcing potential

Among all S_3 -invariant polynomials of degree ≤ 4 in χ (a 10-dimensional space generated by power sums p_1, p_2, p_3), the requirement that the cone $\{n_1 = n_2\}$ be a critical manifold selects a 4-parameter family; requiring it to be a *minimum at every scale* further eliminates the cubic invariant $C_3 = p_1 p_2 - \frac{2}{9} p_1^3$ (whose transverse curvature $\propto -4\sqrt{3} \delta s^3$ changes sign with scale) and fixes the angular part uniquely:

$$V_{\text{eq}} = (n_1 - n_2)^2 = \frac{4}{9} (p_1^2 - \frac{3}{2} p_2)^2. \quad (3)$$

The minimum has two flat directions (overall scale; the angle ϕ inside the doublet): the potential fixes Q but *not* the individual masses, whose selection requires p_3 -type terms as in yukawaon models [7]. C_3 remains available as the leading controlled Koide-breaking operator, with a characteristic scale dependence $\delta Q \propto N^{-1/2}$.

III. RENORMALIZATION-GROUP PROPERTIES AND SCHEME DEPENDENCE

Q is exactly invariant under universal rescalings $m_i \rightarrow \lambda m_i$, hence an RG quasi-invariant: the sensitivity vector

$$\frac{\partial \ln Q}{\partial \ln m_i} = w_i^S - w_i^R, \quad \sum_i (w_i^S - w_i^R) = 0, \quad (4)$$

with $w_i^S = m_i / \sum m$, $w_i^R = \sqrt{m_i} / \sum \sqrt{m}$, is purely doublet. The flavor non-universal part of the on-shell/ $\overline{\text{MS}}$ conversion gives a closed-form drift,

$$\delta Q = Q \frac{3\alpha}{2\pi} \sum_i (w_i^S - w_i^R) \ln m_i = +0.00115, \quad (5)$$

reproducing the running-mass value $Q(M_Z) \simeq 0.6678$ [8]. Above M_Z the non-universal Yukawa running freezes Q to $O(10^{-6})$ up to the GUT scale. Thus the exact relation holds in the unique physical (on-shell) scheme and drifts by $O(\alpha)$ in conventional ones - the behavior required of a geometric relation among physical masses; gauged-family cancellation mechanisms [9] provide a dynamical realization. The scheme spread for the (c, b, t) triplet, evaluated with 5-loop running and 4-loop mass conversions using RunDec/CRunDec [10–12], ranges from $Q = 0.663$ ($\overline{\text{MS}}$ top at its own scale) to $Q = 0.720$ (all masses at M_Z): the definitional ambiguity is commensurate with the deviation itself, i.e. $O(\alpha_s)$ times the logarithmic hierarchy, exactly as the covariance interpretation requires (Table I).

TABLE I. Scheme and scale dependence of Q for the (c, b, t) triplet. Masses in GeV. Running and decoupling at five loops and the pole - $\overline{\text{MS}}$ top-mass conversion at four loops ($n_f = 6$) are performed with RunDec/CRunDec [10–12], with $\alpha_s(M_Z) = 0.1180$, inputs $m_c(m_c) = 1.27$, $m_b(m_b) = 4.18$, $m_t^{\text{pole}} = 172.57$ GeV, the charm mass evolved through the $n_f = 4 \rightarrow 5$ threshold at $\mu = m_b$ and the top mass evolved in $n_f = 6$. The definitional spread $\Delta Q \simeq 0.058$ is commensurate with the deviation from $2/3$ itself, i.e. $O(\alpha_s)$.

Convention	m_c	m_b	m_t	Q	$Q - \frac{2}{3}$
$m_q(m_q)$, pole top	1.270	4.180	172.57	0.6694	+0.0027
$m_q(m_q)$, $\overline{\text{MS}}$ top	1.270	4.180	162.07	0.6625	-0.0042
all at $\mu = M_Z$	0.628	2.863	170.60	0.7204	+0.0537
all at $\mu = 10$ GeV	0.797	3.636	214.00	0.7191	+0.0525

TABLE II. Koide ratio across fermion triplets. Charged-lepton and top masses are pole masses, quark masses $\overline{\text{MS}}$ at their conventional scales (PDG [3]); uncertainties from Monte Carlo propagation of the mass errors (two-sided Gaussian for asymmetric errors). The covariance proxy is α_s evaluated at the lightest member of the triplet (frozen below 1 GeV; zero for the colorless lepton triplet). The row $(c, b, t_{\overline{\text{MS}}})$ illustrates the top-scheme sensitivity discussed in Sec. III. Deviations $|Q - \frac{2}{3}|$ grow monotonically with the covariance class over five orders of magnitude: class means are 3×10^{-6} (no QCD), 2.7×10^{-3} (perturbative), 0.139 (nonperturbative); Spearman $\rho_s = +0.845$ ($p = 0.034$, pole top, $n = 6$).

Triplet	Q	σ_Q	$\frac{Q-2/3}{\sigma_Q}$	θ [deg]	α_s	$ Q - \frac{2}{3} $
(e, μ, τ)	0.66666	0.00001	-0.4	44.9999	0	3×10^{-6}
(c, b, t)	0.66937	0.00079	+3.4	45.116	0.319	0.0027
$(c, b, t_{\overline{\text{MS}}})$	0.66275	0.00080	-4.9	44.831	0.319	0.0039
(d, s, b)	0.73143	0.00509	+12.7	47.540	0.358	0.0648
(u, c, t)	0.84894	0.00106	+171	51.199	0.358	0.1823
(s, c, b)	0.45851	0.00225	-92.7	31.501	0.358	0.2082
(u, d, s)	0.56704	0.01205	-8.3	39.941	0.358	0.0996

IV. COVARIANCE TEST ACROSS FERMION TRIPLETS

A. Monotonicity with the coupling

Deviations $|Q - \frac{2}{3}|$ grow monotonically with the QCD covariance class over five orders of magnitude (Table II): $10^{-5} \rightarrow 3 \times 10^{-3} \rightarrow 0.06 - 0.21$ (Spearman $\rho_s = +0.845$, $p = 0.034$; Fig. 1), stable under five alternative covariance proxies (Sec. VII).

B. First-order formula $\delta Q(\Sigma)$

Modelling interactions as off-diagonal covariance $C_{ij} = \rho \sqrt{m_i m_j}$ with observed masses as eigenvalues gives, at

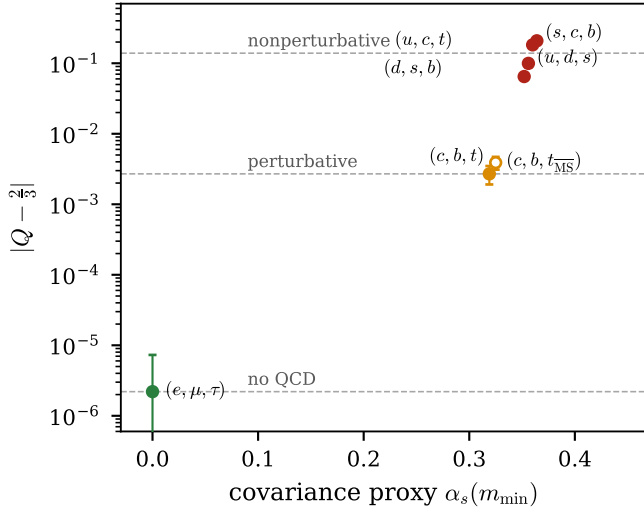


FIG. 1. Deviation $|Q - \frac{2}{3}|$ versus the covariance proxy $\alpha_s(m_{\min})$ for the triplets of Table II (points at the frozen value are horizontally offset for clarity; the open symbol shows the $\overline{\text{MS}}$ -top variant). Dashed lines mark the class means; the deviation grows monotonically with the covariance class over five orders of magnitude.

leading order,

$$\delta \ln Q = \rho^2 B_E, \quad B_E = \sum_i (w_i^S - w_i^R) \sum_{j \neq i} \frac{m_j}{m_i - m_j} > 0, \quad (6)$$

predicting $Q > \frac{2}{3}$ for eigen-triplets of a single mass matrix. All three family-aligned triplets lie above and both cross-family triplets below the cone (5/5 signs), the latter consistent with an additive flavor-blind contamination $\delta Q = \epsilon B_A$, $B_A < 0$, with $B_A = (3 - Q R S_{-1/2})/R^2$, where $R = \sum_i \sqrt{m_i}$ and $S_{-1/2} = \sum_i m_i^{-1/2}$, which is negative for hierarchical triplets containing a light member. The fitted correlation ladder, $\rho \simeq 0.001 (e, \mu, \tau) \rightarrow 0.137 (c, b, t) \rightarrow 0.76 (d, s, b) \rightarrow \text{saturation } (\rho^2 > 1) (u, c, t)$, is monotone in the coupling, saturates exactly at the nonperturbative boundary (the hard bound $\rho^2 \leq 1$ acting as an internal falsifier), and in the perturbative regime matches $\rho(c, b, t) \approx C_F \alpha_s(m_c)/\pi$ at the $\sim 20\%$ level.

V. NEUTRINO SECTOR: B_3 CLASSIFICATION AND PREDICTIONS

Allowing $\chi_i < 0$ extends the symmetry to the hyperoctahedral group $B_3 = S_3 \ltimes \mathbb{Z}_2^3$; modulo the global sign, four sign classes remain. With measured Δm^2 [13], the equipartition cone intersects *exactly one* orbit in each mass ordering - a single reflection on the lightest mode - and each in exactly one point (excluded classes miss by 12% to a factor ≥ 2.5). The phase π of [14] is therefore not a choice but the unique possibility, with a topological origin: only the lightest-mode reflection interpolates

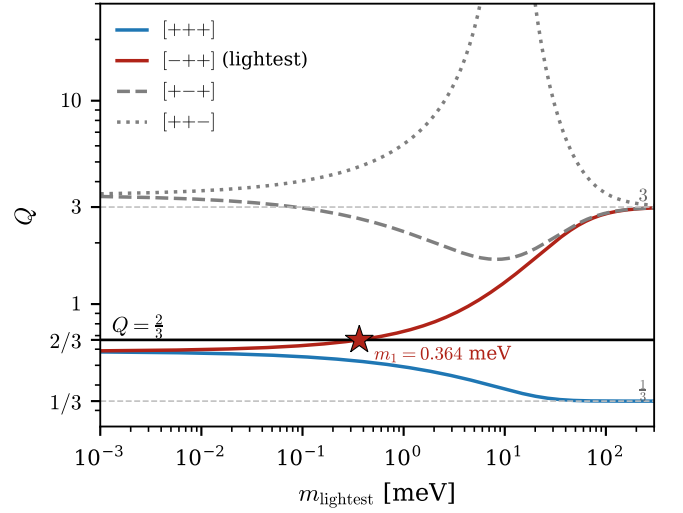


FIG. 2. The Koide ratio Q as a function of the lightest neutrino mass for the four B_3 sign classes (normal ordering, measured Δm^2). Only the lightest-mode reflection $[-++]$ interpolates between the degenerate limits $Q \rightarrow \frac{1}{3}$ and $Q \rightarrow 3$ and crosses the equipartition value $\frac{2}{3}$, at the unique point $m_1 = 0.364$ meV (star); the divergence of the $[-++]$ and $[+-+]$ curves marks the vanishing of $\sum_i s_i \sqrt{m_i}$. The inverted ordering is qualitatively identical.

TABLE III. Neutrino observables predicted by the equipartition cone with a single reflection (π phase) on the lightest mode, for both mass orderings. Inputs: $\Delta m_{21}^2 = 7.49(19) \times 10^{-5} \text{ eV}^2$, $\Delta m_{31}^2 = 2.513(21) \times 10^{-3} \text{ eV}^2$ (NO), $|\Delta m_{32}^2| = 2.484(21) \times 10^{-3} \text{ eV}^2$ (IO), $\sin^2 \theta_{12} = 0.308(12)$, $\sin^2 \theta_{13} = 0.02215(58)$ [13]; uncertainties from Monte Carlo propagation. The $m_{\beta\beta}$ range spans free Majorana phases; the fixed-phase value assumes the sign of $\sqrt{m_{\text{lightest}}}$ in the cone condition is a physical Majorana phase $\alpha = \pi$ (a speculative but distinctive refinement).

Observable	Normal ordering	Inverted ordering
(m_1, m_2, m_3) [meV]	(0.364, 8.662, 50.13)	(49.17, 49.93, 2.917)
Σm_ν [meV]	59.16 ± 0.24	102.0 ± 0.4
m_β [meV]	8.85 ± 0.10	48.9 ± 0.2
$m_{\beta\beta}$ (free phases) [meV]	1.25 - 3.97	18.2 - 48.4
$m_{\beta\beta}$ (fixed phase) [meV]	3.47 ± 0.11	48.2 ± 0.2

continuously between the degenerate limits $Q \rightarrow \frac{1}{3}$ (no reflection) and $Q \rightarrow 3$ (one reflection), and must therefore cross $\frac{2}{3}$ (Fig. 2).

The predictions of Table III are confronted by experiments that are already running or under construction. JUNO, taking data since 2025 with its first oscillation results published [15], will determine the mass ordering, eliminating one branch outright; its updated $\Delta m_{21}^2 = 7.50(12) \times 10^{-5} \text{ eV}^2$ shifts the predicted sum by only 0.006 meV, well within the quoted uncertainty. Cosmology bounds the sum: current ΛCDM analyses give $\Sigma m_\nu \lesssim 64 \text{ meV}$ [16], already pressuring the IO

branch and closing in on the NO value, which lies only 0.4 meV above the oscillation minimum. Direct kinematic searches cannot reach the NO prediction ($m_\beta = 8.85$ meV versus the KATRIN bound of 0.45 eV [17] and the Project 8 target of ~ 40 meV [18]), while the IO value $m_\beta = 48.9$ meV would be visible to Project 8. For $0\nu\beta\beta$, the NO band 1.3 - 4.0 meV lies below the sensitivity of the next generation (LEGEND-1000: $\sim 9 - 21$ meV [19]; nEXO: $\sim 5 - 27$ meV [20]), whereas the IO band 18-48 meV sits squarely inside it; the current KamLAND-Zen limit is $m_{\beta\beta} < 28 - 122$ meV [21]. The construction is therefore doubly falsifiable: an inverted ordering from JUNO combined with a null $0\nu\beta\beta$ result at $\lesssim 18$ meV, a cosmological measurement of Σm_ν below ~ 58.8 meV, or *any* positive $0\nu\beta\beta$ signal with $m_{\beta\beta} > 5$ meV under a confirmed normal ordering, would each exclude it.

VI. STATISTICAL SIGNIFICANCE

The local p -value is estimated in two independent ways. First, by conditional tuning: at fixed m_e and m_τ there is a unique m_μ satisfying $Q = \frac{2}{3}$, and with a log-uniform prior $p = 2 \delta \ln m_\mu / \ln(m_\tau/m_e)$, where the allowed window $\delta \ln m_\mu$ follows from the agreement band $|Q_0 - \frac{2}{3}| + \sigma_Q$ divided by the spurion sensitivity $|Q(w^S - w^R)_\mu|$ of Eq. (4). Second, by Monte Carlo null densities: sampling mass triplets under three priors (log-uniform masses, log-uniform ratios, and the observed hierarchy with a factor-3 log-normal spread) yields $f_Q(\frac{2}{3})$ of order unity in all cases, and $p_{\text{loc}} = 2 f_Q(\frac{2}{3})$ tol. Both routes agree: $p_{\text{loc}} \simeq 2 \times 10^{-5}$ (4.3σ). A conservative look-elsewhere accounting (counting coprime fractions k/n with $n \leq 6-20$ in the allowed interval $(\frac{1}{3}, 1)$, times a structural multiplier of up to 10 for alternative functional forms: 7-850 trials) leaves $2.4-3.8\sigma$ globally. Independently, the 1981 out-of-sample test is trials-free: with $m_\tau = 1784.2 \pm 3.2$ MeV the relation was $+2.3\sigma$ away and effectively predicted $m_\tau = 1776.97$ MeV; subsequent measurements landed within a ± 0.13 MeV band (the present agreement band translated into m_τ) out of a plausible ± 9.6 MeV ($\pm 3\sigma_{1981}$) range, giving $p_{\text{oos}} = 0.013$ with no trials penalty, since the relation was selected before the measurements.

VII. EXTENSIONS AND ROBUSTNESS

Charge-transverse generations form a ladder $Q_{(u,d,e)} = 0.389 \rightarrow Q_{(c,s,\mu)} = 0.476 \rightarrow Q_{(t,b,\tau)} = 0.655$, a third independent manifestation of the covariance hierarchy; sets with $N > 3$, considered in the spirit of extended sums over larger fermion collections [22], prefer neither $\frac{2}{3}$ nor the S_N value $2/N$ - equipartition is a property of S_3 family triplets. Belle II precision $\sigma_{m_\tau} \rightarrow 0.02$ MeV will probe the predicted accuracy floor $|Q - \frac{2}{3}| \sim 10^{-6}$

from electroweak covariance [4].

VIII. DISCUSSION AND OUTLOOK

The equipartition formulation supplies what the Koide relation has lacked: a language (irreducible representations of the family group), a mechanism (the unique enforcing potential V_{eq}), and a family of quantitative consequences that go beyond the original single number - the covariance ladder of Table II, the sign dichotomy of Eq. (6), and the neutrino observables of Table III. It does not, by itself, constitute a first-principles derivation: the choice of $\chi = \sqrt{m}$ as the fundamental variable is motivated by the variance-like additivity of mass (amplitudes versus probabilities; field values versus energies) but remains a postulate, and the microscopic origin of the covariance structure is modelled, not derived.

Two systematic issues deserve emphasis. First, the scheme question: the exact relation selects the on-shell scheme, and the verdicts of the covariance models inherit this choice - with an $\overline{\text{MS}}$ top the (c,b,t) deviation changes sign (Table I). Within the present interpretation this is consistent rather than problematic - a renormalization scheme is precisely a bookkeeping choice of how much self-covariance is absorbed into the mass definition, and the physical (on-shell) masses are the natural coordinates of a geometric relation - but a complete theory should derive this, as the gauged-family cancellation of [9] does dynamically. Second, statistics: six triplets and one proxy family are a directional test, not a discovery; the strength of the case rests on the convergence of independent pillars (precision, RG structure, covariance monotonicity with correct signs, the 1981 prospective hit) rather than on any single one.

The falsification schedule is short. JUNO will fix the mass ordering within this decade [15]; DESI and Euclid will measure or bound Σm_ν at the few-meV level near the predicted 59.2 meV [16]; LEGEND-1000 and nEXO will sweep the entire IO band of $m_{\beta\beta}$ [19, 20]; and Belle II will sharpen m_τ to the point where the predicted electroweak accuracy floor $|Q - \frac{2}{3}| \sim 10^{-6}$ becomes observable [4]. Each outcome either removes the construction or promotes it from an interpretation to a constraint that any theory of flavor must accommodate. Natural theoretical continuations include the yukawaon-type selection of the doublet angle ϕ (fixing individual masses) [7], the relative orientation of the charged-lepton and neutrino orbits as a possible origin of the PMNS structure, and a derivation of the covariance matrix Σ from the underlying gauge dynamics.

DATA AVAILABILITY

All analysis scripts reproducing the tables, figures and statistical results of this work are publicly available at 10.5281/zenodo.21167247.

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